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Simplex Method

The simplex method is a general **algebraic method** which can be used to solve LP problems with a very large number of variables and constraints. The method provides an algorithm starts with basic feasible solution, in general, from **origin**, and then for finding a neighbouring extreme point.

The simplex algorithm works with **standard form** of the problem. It starts from a basic feasible solution on the boundary of the polyhedron representing the feasible region of the problem and hops to an adjacent basic feasible solution improving the value of the objective of the problem in the process. The algorithm *stops* when no further improvement is possible. The algorithm also detects infeasibility (no value satisfies the constraints) and unboundedness where the objective can be made arbitrarily large.

Standard Form

Before the simplex algorithm can be used to solve an LP, the LP must be converted into an equivalent problem in which *all constraints are equations* and *all variables are nonnegative*

An LP in this form is said to be in **standard form**.

$$\begin{aligned} \min \quad & c^T x \\ s.t \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

• converting inequalities into equations with **nonnegative r.h.s**.

- For $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$, introduce slack variables s_i (it is numerically equal to the difference between the r.h.s and l.h.s representing the worse involved in the phase of the system by the constraint).
الفرق بين الطرفين مثلا الايمن 10 و الايسر 6 فالفرق الغير مستعمل هو (s = 4). 6+4 = 10

$$\sum_{j=1}^{n} a_{ij}x_j + s_i = b_i, \qquad s_i \geq 0$$

- For $\sum_{j=1}^{n} a_{ij}x_j \geq b_i$, introduce surplus variable s_i (it is numerically equal to the difference between the l.h.s and r.h.s representing the excess input in the phase of the system by the constraint).
هنا فرق التجاوزحيث الطرف الايمن مثلا 5 والايسر 15 فالفرق هو (s = 10) 15-10=5

$$\sum_{j=1}^{n} a_{ij}x_j - s_i = b_i, \quad s_i \geq 0$$

Example: Write the following model as a standard form

$$\begin{aligned} \max \ z &= 2x_1 + 3x_2 \\ 2x_1 + x_2 &\leq 4 \\ \text{s.t} \quad x_1 + 2x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution:

$$\begin{aligned} \max \ z &= 2x_1 + 3x_2 \\ 2x_1 + x_2 + s_1 &= 4 \\ x_1 + 2x_2 + s_2 &= 5 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$

Q) Write the following model as a standard form

$$\begin{aligned} \max \ z &= 6x_1 + 8x_2 \\ \text{s.t} \quad 4x_1 + x_2 &\leq 8 \\ x_1 + 4x_2 &\geq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Basic Solution for standard form

If there are (m) equality constraints in (n) variables with $n > m$, **basic solutions (**correspond to the corner points in the graphical solution space**)** can be obtained by setting any of the (n-m) variables to zero, and solving the m equations for the remaining m variables, provided the resulting solution is unique. We find all the basic solutions and the optimum is the feasible basic solution with the best objective value.

اذا كانت $n > m$ سيكون لدينا اكثر من حل لهذا يجب ان يكون $n = m$ و ذلك بوضع المتغيرات الاضافية تساوي صفر

This means that the maximum number of corner points is

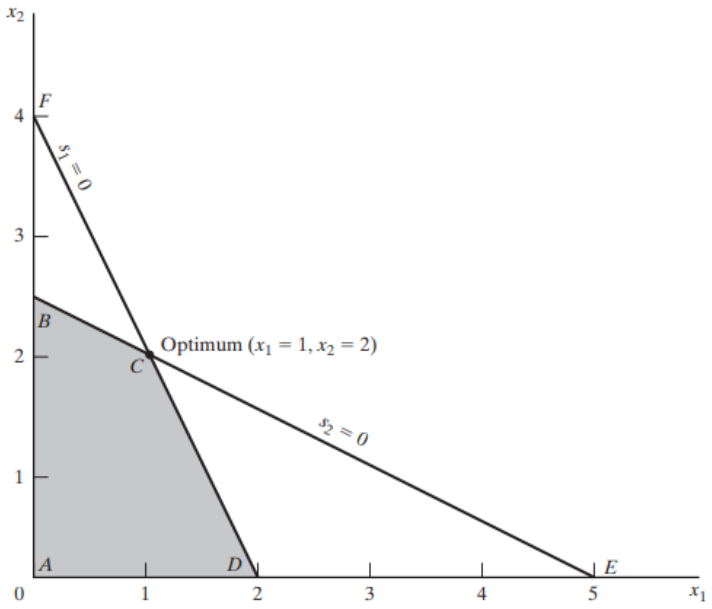
$$C_m^n = \frac{n!}{m!(n-m)!}$$

For example if n=10 and m=5 then $\frac{10!}{5!(10-5)!} = 252$ basic solutions

Note: The basic solution may be feasible or not. Simplex method however, works only with *basic feasible solutions*.

Example: Consider the following LP with two variables: (page 102 Hamdy Taha)

$$\begin{aligned} \text{Maximize } & z = 2x_1 + 3x_2 \\ & 2x_1 + x_2 \leq 4 \\ \text{s.t} \quad & x_1 + 2x_2 \leq 5 \\ & x_1, x_2 \geq 0 \end{aligned}$$



Algebraically, rewrite the model as a standard form the solution space of the LP is represented by the following $m = 2$ equations and $n = 4$ variables:

$$\begin{aligned} 2x_1 + x_2 + s_1 &= 4 \\ x_1 + 2x_2 + s_2 &= 5 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$

The basic solutions are determined by setting $n - m\ (= 4 - 2 = 2)$ variables equal to zero and solving the equations for the remaining $m\ (= 2)$ variables.

For example, if we set $x_1 = 0$ and $x_2 = 0$, the equations provide the unique basic solution

$$s_1 = 4, s_2 = 5$$

This solution corresponds to point A in the Figure (convince yourself that $s_1 = 4$ and $s_2 = 5$ at point A). Another point can be determined by setting $s_1 = 0$ and $s_2 = 0$ then solving the resulting two equations

$$\begin{aligned} 2x_1 + x_2 &= 4 \\ x_1 + 2x_2 &= 5 \end{aligned}$$

The associated basic solution is ($x_1 = 1$, $x_2 = 2$), or point C.

In this example, the (maximum) number of corner points is $\frac{4!}{2!\,2!} = 6$. Looking at the Figure, we can spot the four corner points A, B, C, and D. So, where are the remaining two? In fact, points E and F also are corner points. But, they are **infeasible**, and, hence, are not candidates for the optimum.

To complete the algebraic solution, the zero n - m variables are known as **nonbasic** variables. The remaining m variables are called **basic variables**, and their solution (obtained by solving the m equations) is referred to as **basic solution**. The following table provides all the basic and nonbasic solutions.

Nonbasic (zero) variables	Basic variables	Basic solution	Associated corner point	Feasible?	Objective value, z
(x_1, x_2)	(s_1, s_2)	(4, 5)	*A*	Yes	0
(x_1, s_1)	(x_2, s_2)	(4, −3)	*F*	No	–
(x_1, s_2)	(x_2, s_1)	(2.5, 1.5)	*B*	Yes	7.5
(x_2, s_1)	(x_1, s_2)	(2, 3)	*D*	Yes	4
(x_2, s_2)	(x_1, s_1)	(5, –6)	*E*	No	–
(s_1, s_2)	**(x_1, x_2)**	**(1, 2)**	***C***	**Yes**	**8 (optimum)**

Note: Rather than enumerating all the basic solutions (corner points) of the LP problem (as we did in the example above), the simplex method investigates only a "select few" of these solutions

H.W: Find the basic feasible solutions for the following linear programming problems.

$$max \quad z = 3x_1 + 2x_2$$

$$\begin{aligned} subject\ to \quad & 2x_1 + x_2 \le 100 \\ & x_1 + x_2 \le 80 \\ & x_1 \le 40 \\ & x_1 \ge 0,\ x_2 \ge 0 \end{aligned}$$

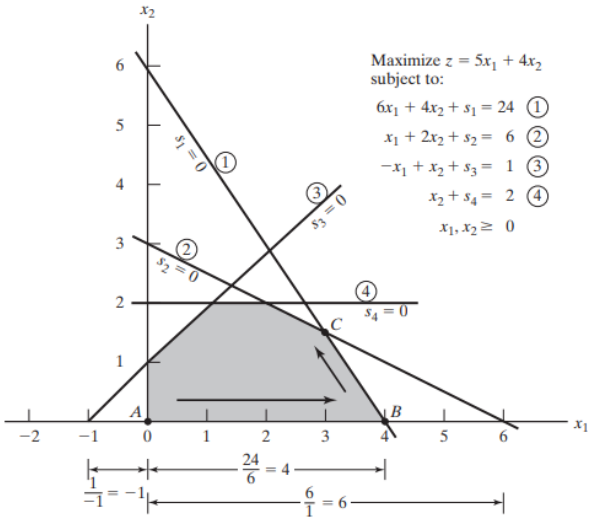
$$max \quad z = 5x_1 + 6x_2$$

$$\begin{aligned} subject\ to \quad & x_1 + x_2 = 5 \\ & x_1 \ge 2 \\ & x_2 \le 4 \\ & x_1 \ge 0,\ x_2 \ge 0 \end{aligned}$$

Simplex Method

Example: Consider the Reddy Mikks model again expressed in equation form.

$$\begin{aligned} \max \quad z = {} & 5x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4 \\ & 6x_1 + 4x_2 + s_1 = 24 \\ & x_1 + 2x_2 + s_2 = 6 \\ & -x_1 + x_2 + s_3 = 1 \\ & x_2 + s_4 = 2 \\ & x_1, x_2, s_1, s_2, s_3, s_4 \geq 0 \end{aligned}$$



The variables $s_1, s_2, s_3, and\ s_4$ are the slacks. Next, we write the objective equation as

$$z - 5x_1 - 4x_2 - 0s_1 - 0s_2 - 0s_3 - 0s_4 = 0$$

The simplex tableau can be represented as follows:

Basic	x_1	x_2	s_1	s_2	s_3	s_4	Solution	
z	−5	−4	0	0	0	0	0	z-row
s_1	6	4	1	0	0	0	24	s_1-row
s_2	1	2	0	1	0	0	6	s_2-row
s_3	−1	1	0	0	1	0	1	s_3-row
s_4	0	1	0	0	0	1	2	s_4-row

The solution starts at the origin [$(x_1, x_2) = (0, 0)$], thus defining (x_1, x_2) as the **nonbasic variables** and (s_1, s_2, s_3, s_4) as the **basic variables**.

By the rule of **simplex optimality condition, w**e choose the smallest negative coefficient number in the z-row to determine the **entering variable** to enter the

solution, so we choose x_1, it becomes basic variable. In this case one of the basic variables should leave and become nonbasic variable.

نختار x_1 متغير داخل لانه يملك اعلى معامل في دالة الهدف و عليه سيزيد دالة الهدف

We compute the ratios of the right hand side of the equations (Solution column) to the corresponding (strictly) positive constraint coefficients under the entering variable, x_1, We can see that the value of x_1 must be increased to the smallest nonnegative intercept with the x_1-axis (= 4) to reach corner point B. Any increase beyond B is infeasible. At point B, the current basic variable s_1 associated with constraint 1 assumes a zero value and becomes the **leaving variable**.

نختار اصغر نسبة موجبة و هي 4. واي رقم اكبر منه سيكون في المنطقة الغير واضحة مثلا النسبة 6 ستكون بالخارج. انظر الى الرسم (x_1 -axis (= 6

The new solution point B is determined by "swapping" the entering variable x_1 and the leaving variable s_1 in the simplex tableau to yield

min ratio = {24/6, 6/1, 1/-1, 2/0} = 4

Note: ratio computations is referred to as the **simplex feasibility condition** because it guarantees the feasibility of the new solution.

		Enter ↓							
	Basic	x_1	x_2	s_1	s_2	s_3	s_4	Solution	
	z	−5	−4	0	0	0	0	0	
Leave ←	s_1	6	4	1	0	0	0	24	row1 Pivot row
	s_2	1	2	0	1	0	0	6	row2
	s_3	−1	1	0	0	1	0	1	row3
	s_4	0	1	0	0	0	1	2	row4
		Pivot column							

We call the entering variable column as the **pivot column** and the leaving variable as the **pivot row**. We use the Gauss-Jordan computations

1. New row1 = Current row1 ÷ 6
   = 1/6 (6 4 1 0 0 0 24) = (1 2/3 1/6 0 0 0 4)

2. New z-row = Current Z-row + 5 × New row1
   = (-5 -4 0 0 0 0 0) + 5 × (1 2/3 1/6 0 0 0 4)
   = (0 -2/3 5/6 0 0 0 20)

3. New row2 = Current row2 – 1 × New row1
   = (1 2 0 1 0 0 6) – 1 × (1 2/3 1/6 0 0 0 4)
   = (0 4/3 -1/6 1 0 0 2)

4. New row3 = Current row3 + New row1
   = (-1 1 0 0 1 0 1) + (1 2/3 1/6 0 0 0 4)
   = (0 5/3 1/6 0 1 0 5)

5. New row4 = (0 1 0 0 0 1 2)

Nonbasic (zero) variables at B: (s_1, x_2), Basic variables at B: (x_1, s_2, s_3, s_4)

	Basic	x_1	x_2 ↓	s_1	s_2	s_3	s_4	Solution
	z	0	$-\frac{2}{3}$	$\frac{5}{6}$	0	0	0	20
	x_1	1	$\frac{2}{3}$	$\frac{1}{6}$	0	0	0	4
←	s_2	0	$\frac{4}{3}$	$-\frac{1}{6}$	1	0	0	2
	s_3	0	$\frac{5}{3}$	$\frac{1}{6}$	0	1	0	5
	s_4	0	1	0	0	0	1	2

In the last tableau, the optimality condition shows that x_2 (with the most negative z-row coefficient) is the entering variable. Then feasibility condition produces: min ratio = {4÷2/3, 2÷4/3, 5÷5/3, 2÷1} = 1.5

We use the Gauss-Jordan computations

1. New row2 = Current row2 × 3/4
   = 3/4 (0 4/3 -1/6 1 0 0 2) = (0 1 -1/8 3/4 0 0 3/2)

2. New z-row = Current z-row + 2/3 × New row2
   = (0 -2/3 5/6 0 0 0 20) + 2/3 × (0 1 -1/8 3/4 0 0 3/2)
   = (0 0 3/4 1/2 0 0 21)

3. New row1 = Current row1 – 2/3 × New row2
   = (1 2/3 1/6 0 0 0 4) – 2/3 × (0 1 -1/8 3/4 0 0 3/2)
   = (1 0 1/4 -1/2 0 0 3)

4. New row3 = Current row3 - 5/3 × New row2
   = (0 5/3 1/6 0 1 0 5) - 5/3 × (0 1 -1/8 3/4 0 0 3/2)
   = (0 0 3/8 -5/4 1 0 5/2)

5. New row4 = Current row4 – 1 × New row2
   = (0 1 0 0 0 1 2) – 1 × (0 1 -1/8 3/4 0 0 3/2)
   = (0 0 1/8 -3/4 0 1 1/2)

Basic	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21
x_1	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	3
x_2	0	1	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
s_3	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
s_4	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$

لا يوجد معاملات سالبة في row-z عليه الحل كامل z = 5×3 + 4×3/2 = 21

Decision variable	Optimum value	Recommendation
x_1	3	Produce 3 tons of exterior paint daily
x_2	$\frac{3}{2}$	Produce 1.5 tons of interior paint daily
z	21	Daily profit is $21,000

Procedure: (I-phase) Simplex Algorithm

```
begin (for max problem with z − Σ_j c_j x_j = 0 being the cost row)
    OPT := "NO"
    UNBOUNDED := "NO"
    while OPT = "NO" and UNBOUNDED = "NO", do
        if c_j ≥ 0, ∀j then OPT := "YES"
        else
            begin
                choose j such that c_j < 0 is the most negative
                if y_ij ≤ 0, ∀i then UNBOUNDED := "YES"
                else
                    find θ_0 = min{y_i0 / y_ij, ∀i and y_ij ≥ 0}
                    denote the index by k and pivot on k
                endif
            end
        endif
    endwhile
end
```

H.W: solve the following linear programming problems by simplex method

$$\begin{aligned} \max \quad & 8\,x_1 + 5x_2 \\ s.t \quad & x_1 + x_2 \le 6 \\ & 9x_1 + 5x_2 \le 45 \\ & x_1, \quad x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} max \quad & z = \ 3\,x_1 + 2\,x_2 \\ subject\ to \quad & 2\,x_1 + \ x_2 \le 100 \\ & x_1 + \ x_2 \le 80 \\ & x_1 \qquad \le 40 \\ & x_1 \ge 0,\ x_2 \ge 0 \end{aligned}$$

Note: In minimization problems, the optimality condition calls for selecting the entering variable as the nonbasic variable with the most positive objective coefficient in the z-row, the exact opposite rule of the maximization case. This follows because max z is equivalent to min (–z). As for the feasibility condition for selecting the leaving variable, the rule remains unchanged.